The Simplest DSGE Model

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1. Introduction

What is a DSGE macroeconomic model?

- **D** for *dynamics*: the economy evolves over time; economic decisions are made over time
- **S** for *stochastic*: the economy is exposed to external shocks that can not be anticipated or forecasted
- **G** for *general*: considers all markets that are important for the functioning of a modern economy
- **E** for *equilibrium*: private agents and public decision-making institutions try to do the best they can with all available information (optimal decision making)

How relevant are DSGE models?

- As Stanley Fischer put it here:

> "Let me turn to [...] macroeconomic models and their role in assisting the FOMC's decisionmaking. The Board staff maintains several models; I will focus on the FRB/US model, the best known and most used of the models the Board staff has at its disposal. FRB/US is an *estimated, large-scale, general-equilibrium, New Keynesian model.*"

in: "*I'd rather have Bob Solow than an econometric model, but ...*", Speech by Stanley Fischer, Vice Chair of the Board of Governors of the Federal Reserve System, at the Warwick Economics Summit, 11 February 2017.

The Simplest possible model

- The model is linear
- It has three types of variables:
  - a forward-looking variable/block: y_t
  - a predetermined variable/block: x_t
  - a contemporaneous (or static) variable/block: z_t
- It is an uncoupled model: each variable/block can be solved separately from all other variables/blocks.
- This property means that we can solve the model with pencil and paper.

The three equations

$$
\begin{aligned}
x_t &= \phi + \rho x_{t-1} + \varepsilon_t^x \\
&\varepsilon_t^x \sim \mathcal{N}\left(0, \sigma^2\right) \\
y_t &= \alpha + \beta \mathbb{E}_t y_{t+1} + \theta x_t \\
z_t &= \varphi x_t + \mu y_t
\end{aligned}
$$

- x_t is a backward-looking (or pre-determined) variable
- ε_t^x is a random shock
- y_t is a forward-looking variable
- z_t is a contemporaneous (or static) variable
- $\{\alpha, \beta, \theta, \phi, \rho, \varphi, \mu\}$ are parameters

2. Solution by pencil-and-paper

Pencil & paper solution: backward-looking block

- For simplicity, instead of ε_t^x, we use ε_t to simplify the notation as much as possible.
- We will solve the equation $x_t = \phi + \rho x_{t-1} + \varepsilon_t$ by repeated substitution (iterations) backwards in time.
- Like this, when $n \to \infty$:

$$\underbrace{t \to (t-1)}_{\text{1st iteration}} \to \underbrace{(t-1) \to (t-2)}_{\text{2nd iteration}} \to \underbrace{(t-2) \to (t-3)}_{\text{3rd iteration}} \to \ldots \to \underbrace{(t-(n-1)) \to (t-n)}_{n\text{th iteration}}$$

- The strategy is as follows:
  - Iterate up to the 3rd iteration: see a pattern at this iteration
  - Then, generalize to the nth iteration

$$x_t = \phi + \rho x_{t-1} + \varepsilon_t$$

1st iteration: $t \to t-1$

$$\downarrow \qquad \nwarrow x_{t-1} = \phi + \rho x_{t-2} + \varepsilon_{t-1}$$

Repeated substitution

$$x_t = \phi + \rho\left[\phi + \rho x_{t-2} + \varepsilon_{t-1}\right] + \varepsilon_t$$

$$x_t = \phi + \rho\phi + \rho^2 x_{t-2} + \rho\varepsilon_{t-1} + \varepsilon_t$$

2nd iteration: $t-1 \to t-2$

$$\downarrow \qquad \nwarrow x_{t-2} = \phi + \rho x_{t-3} + \varepsilon_{t-2}$$

$$x_t = \rho^0\phi + \rho^1\phi + \rho^2\phi + \rho^3 x_{t-3} + \rho^2\varepsilon_{t-2} + \rho^1\varepsilon_{t-1} + \rho^0\varepsilon_t$$

$$x_t = \sum_{i=0}^{3-1} \rho^i\phi + \rho^3 x_{t-3} + \sum_{i=0}^{3-1} \rho^i\varepsilon_{t-i}$$

3rd iteration: $t-2 \to t-3$

$$x_t = \sum_{i=0}^{n-1} \rho^i\phi + \rho^n x_{t-n} + \sum_{i=0}^{n-1} \rho^i\varepsilon_{t-i}$$

nth iteration: $\ldots\ t-n$

- To avoid explosive behavior on the solution obtained in the previous slide:

$$x_t = \sum_{i=0}^{n-1} \rho^i \phi + \rho^n x_{t-n} + \sum_{i=0}^{n-1} \rho^i \varepsilon_{t-i}$$

we have to impose the condition: $|\rho| < 1$.

- If $|\rho| < 1$, the solution to this block at the nth iteration (when $n \to \infty$) is:

$$x_t = \sum_{i=0}^{n-1} \rho^i \phi + \sum_{i=0}^{n-1} \rho^i \varepsilon_{t-i} = \frac{\phi}{1-\rho} + \sum_{i=0}^{n-1} \rho^i \varepsilon_{t-i} \tag{1}$$

- Suppose, for simplicity, that $\phi = 0, \rho = 0.5$. So, the solution to x_t will be:

$$x_t = \frac{\phi}{1-\rho} + \sum_{i=0}^{n-1} \rho^i \varepsilon_{t-i} = 0 + \sum_{i=0}^{n-1} 0.5^i \cdot \varepsilon_{t-i} \tag{2}$$

- From equation (2), we can easily conclude that the current value of x_t depends only on the shocks it suffered in the past
- So, what is the value of x_t if it suffered ONE positive shock of $+1$ at period `t-3`?
- It is easy to see that at $t-3$, we get:

$$x_{t-3} = 0.5^0 \times \varepsilon_{t-3} = 1 \times 1 = 1.$$

- As there are no more shocks in this exercise, what is the value of x_{t-2}? Using the original equation ($x_t = \phi + \rho x_{t-1} + \varepsilon_t$), we can obtain:

$$x_{t-2} = 0 + 0.5 \times x_{t-3} + 0 = 0.5 \times 1 = 0.5$$

- Doing the same for x_{t-1} and x_t, we get

$$x_{t-1} = 0.5 \times x_{t-2} + 0 = 0.5 \times 0.5 = 0.25$$
$$x_t = 0.5 \times x_{t-1} + 0 = 0.5 \times 0.25 = 0.125$$

Clarifying the previous slide

In eq (2) we got this result: $x_t = \sum_{i=0}^{n-1} \rho^i \cdot \varepsilon_{t-i}$, or in its expanded form:

$$x_t = \rho^0 \varepsilon_{t-0} + \rho^1 \varepsilon_{t-1} + \rho^2 \varepsilon_{t-2} + \rho^3 \varepsilon_{t-3} \quad ; \quad \rho = 0.5$$

- Suppose the shock occurred 3 periods ago, what is x_t equal to:

$$x_t = 0.5^0 \times 0 + 0.5^1 \times 0 + 0.5^2 \times 0 + 0.5^3 \times 1 = 0.125$$

- Suppose the shock occurred 2 periods ago, what is x_t equal to:

$$x_t = 0.5^0 \times 0 + 0.5^1 \times 0 + 0.5^2 \times 1 = 0.25$$

- Suppose the shock occurred 1 period ago, what is x_t equal to:

$$x_t = 0.5^0 \times 0 + 0.5^1 \times 1 = 0.5$$

- Suppose the shock occurred this period, what is x_t equal to:

$$x_t = 0.5^0 \times 1 = 1$$

Pencil & paper solution: forward-looking block

- We will solve the equation

$$y_t = \alpha + \beta \mathbb{E}_t y_{t+1} + \theta x_t$$

by repeated substitution (iterations) forward in time.

- Like this, when $n \to \infty$:

$$\underbrace{t \to (t+1)}_{\text{1st iteration}} \to$$

$$\underbrace{(t+1) \to (t+2)}_{\text{2nd iteration}} \to$$

$$\underbrace{(t+2) \to (t+3)}_{\text{3rd iteration}} \to \dots$$

$$\underbrace{(t+(n-1)) \to (t+n)}_{n\text{th iteration}}$$

$$y_t = \alpha + \beta \mathbb{E}_t y_{t+1} + \theta x_t$$ 1st iteration: $t \to t+1$

$$\downarrow \qquad \nwarrow \mathbb{E}_t y_{t+1} = \alpha + \beta \mathbb{E}_t y_{t+2} + \theta \mathbb{E}_t x_{t+1}$$ Law Iterat. expectations

$$y_t = \alpha + \beta \left[\alpha + \beta \mathbb{E}_t y_{t+2} + \theta \mathbb{E}_t x_{t+1}\right] + \theta x_t$$

$$y_t = \alpha + \beta\alpha + \beta^2 \mathbb{E}_t y_{t+2} + \beta\theta \mathbb{E}_t x_{t+1} + \theta x_t$$ 2nd iteration: $t+1 \to t+2$

$$\downarrow \qquad \nwarrow \mathbb{E}_t y_{t+2} = \alpha + \beta \mathbb{E}_t y_{t+3} + \theta \mathbb{E}_t x_{t+2}$$

$$y_t = \beta^0\alpha + \beta^1\alpha + \beta^2\alpha + \beta^3 \mathbb{E}_t y_{t+3} + \theta\beta^2 \mathbb{E}_t x_{t+2} + \theta\beta^1 \mathbb{E}_t x_{t+1} + \theta\beta^0 \mathbb{E}_t x_t$$

$$y_t = \sum_{i=0}^{3-1} \beta^i \alpha + \beta^3 \mathbb{E}_t y_{t+3} + \sum_{i=0}^{3-1} \theta\beta^i \mathbb{E}_t x_{t+i}$$ 3rd iteration: $t+2 \to t+3$

$$y_t = \sum_{i=0}^{n-1} \beta^i \alpha + \beta^n \mathbb{E}_t y_{t+n} + \sum_{i=0}^{n-1} \theta\beta^i \mathbb{E}_t x_{t+i}$$ nth iteration: $\ldots \to t+n$

- To avoid explosive behavior on the solution

$$y_t = \sum_{i=0}^{n-1} \beta^i \alpha + \beta^n \mathbb{E}_t y_{t+n} + \sum_{i=0}^{n-1} \theta \beta^i \mathbb{E}_t x_{t+i}$$

we have to impose the condition: $|\beta| < 1$.

- We get the solution to this block at the nth iteration as $n \to \infty$:

$$y_t = \sum_{i=0}^{n-1} \alpha \beta^i + \sum_{i=0}^{n-1} \theta \beta^i \mathbb{E}_t x_{t+i} \tag{3}$$

- The solution to eq. (3), can be written as:

$$y_t = \frac{\alpha}{1-\beta} + \sum_{i=0}^{n-1} \theta \beta^i \mathbb{E}_t x_{t+i} \tag{4}$$

Unconditional vs conditional expectations

- The solution to eq.(4) depends on the type of information we may have about the observations of x_t over time.
- That is: what is the value of $\mathbb{E}_t x_{t+i}$ in eq. (4)?
- It depends on whether we compute the unconditional mean of x_{t+i} , or its conditional mean.
- The unconditional mean is just the deterministic value of its steady state: $\overline{x}$.
- The conditional mean is computed on the basis that we know the value of x_t.
- Next we show how to compute these two expected values.

Unconditional expectations

- Consider that x_t is given by:

$$x_t = \phi + \rho x_{t-1} + \varepsilon_t \quad , \quad \varepsilon_t \sim \mathcal{N}\left(0, \sigma^2\right)$$

- Assuming unconditional expectations, the mean is given by the (deterministic) steady-state value of x_t:

$$x_t = x_{t-1} = \overline{x}$$

- Which leads to

$$\overline{x} = \phi + \rho \overline{x} + 0 \Rightarrow \overline{x} = \frac{\phi}{1-\rho}$$

- Therefore, the expected (unconditional) value of $\mathbb{E}_t x_{t+i}$ is given by:

$$\mathbb{E}_t x_{t+i} = \overline{x} = \frac{\phi}{1-\rho} \tag{5}$$

Conditional expectations

- Apply the expectations operator to $x_t = \phi + \rho x_{t-1} + \varepsilon_t$, up to third iteration:

$$
\begin{aligned}
x_t &= \phi + \rho x_{t-1} + \varepsilon_t \\
\mathbb{E}_t x_{t+1} &= \phi + \rho \mathbb{E}_t x_t + \mathbb{E}_t \varepsilon_{t+1} = \phi + \rho x_t + 0 = \phi + \rho x_t \\
\mathbb{E}_t x_{t+2} &= \phi + \rho \mathbb{E}_t x_{t+1} + \mathbb{E}_t \varepsilon_{t+2} = \phi + \rho \left[\phi + \rho x_t\right] + 0 = \phi + \rho\phi + \rho^2 x_t \\
\mathbb{E}_t x_{t+3} &= \phi + \rho \mathbb{E}_t x_{t+2} + \mathbb{E}_t \varepsilon_{t+3} = \phi + \rho \left[\phi + \rho\phi + \rho^2 x_t\right] + 0 = \underbrace{\phi + \rho\phi + \rho^2\phi}_{=\sum_{k=0}^{3-1} \phi\rho^k} + \rho^3 x_t
\end{aligned}
$$

- Then, generalize to the ith iteration

$$
\mathbb{E}_t x_{t+i} = \sum_{k=0}^{k-1} \phi\rho^k + \rho^i x_t = \frac{\phi}{1-\rho} + \rho^i x_t \tag{6}
$$

- Notice that if we do not know any x_t, then $\mathbb{E}_t x_{t+i} = \dfrac{\phi}{1-\rho}$, which is its unconditional mean.

Forward-looking block solution with conditional expectations

- The solution to the forward-looking block (eq. 4), was given by:

$$y_t = \frac{\alpha}{1-\beta} + \sum_{i=0}^{n-1} \theta \beta^i \mathbb{E}_t x_{t+i} \tag{4a}$$

- From eq. (6), we know that with conditional expectations we have (recall that, for simplicity, we assumed $\phi = 0$). Therefore, we get

$$\mathbb{E}_t x_{t+i} = \frac{\phi}{1-\rho} + \rho^i x_t = 0 + \rho^i x_t \tag{6a}$$

- So, the forward-looking block's solution is given by inserting eq. (6a) into eq. (4a):

$$y_t = \frac{\alpha}{1-\beta} + \sum_{i=0}^{n-1} \theta (\beta\rho)^i x_t = \frac{\alpha}{1-\beta} + \frac{\theta}{1-\rho\beta} x_t. \tag{7}$$

- Given that we know that the parameters are as follows:

$$\alpha = 0 \ , \ \theta = 1 \ , \ \rho = 0.5 \ , \ \beta = 0.75$$

- Inserting these values into eq. (7), we can obtain the numerical solution to the forward-looking block as:

$$y_t = \frac{\alpha}{1-\beta} + \frac{\theta}{1-\rho\beta} x_t$$

$$y_t = \frac{0}{1-0.75} + \frac{1}{1-0.5\times 0.75} x_t = 0 + 1.6 x_t \tag{8}$$

- Therefore, it is easy yo see that

$$\frac{\partial y_t}{\partial x_t} = 1.6$$

- So whenever x_t changes, we know what happens to y_t.

Pencil & paper solution: static block

- The static block is given by the equation:

$$z_t = \varphi x_t + \mu y_t$$

- From the previous slide, we know that

$$x_{t-3} = 1\,, x_{t-2} = 0.5\,, x_{t-1} = 0.25\,, x_t = 0.125.$$

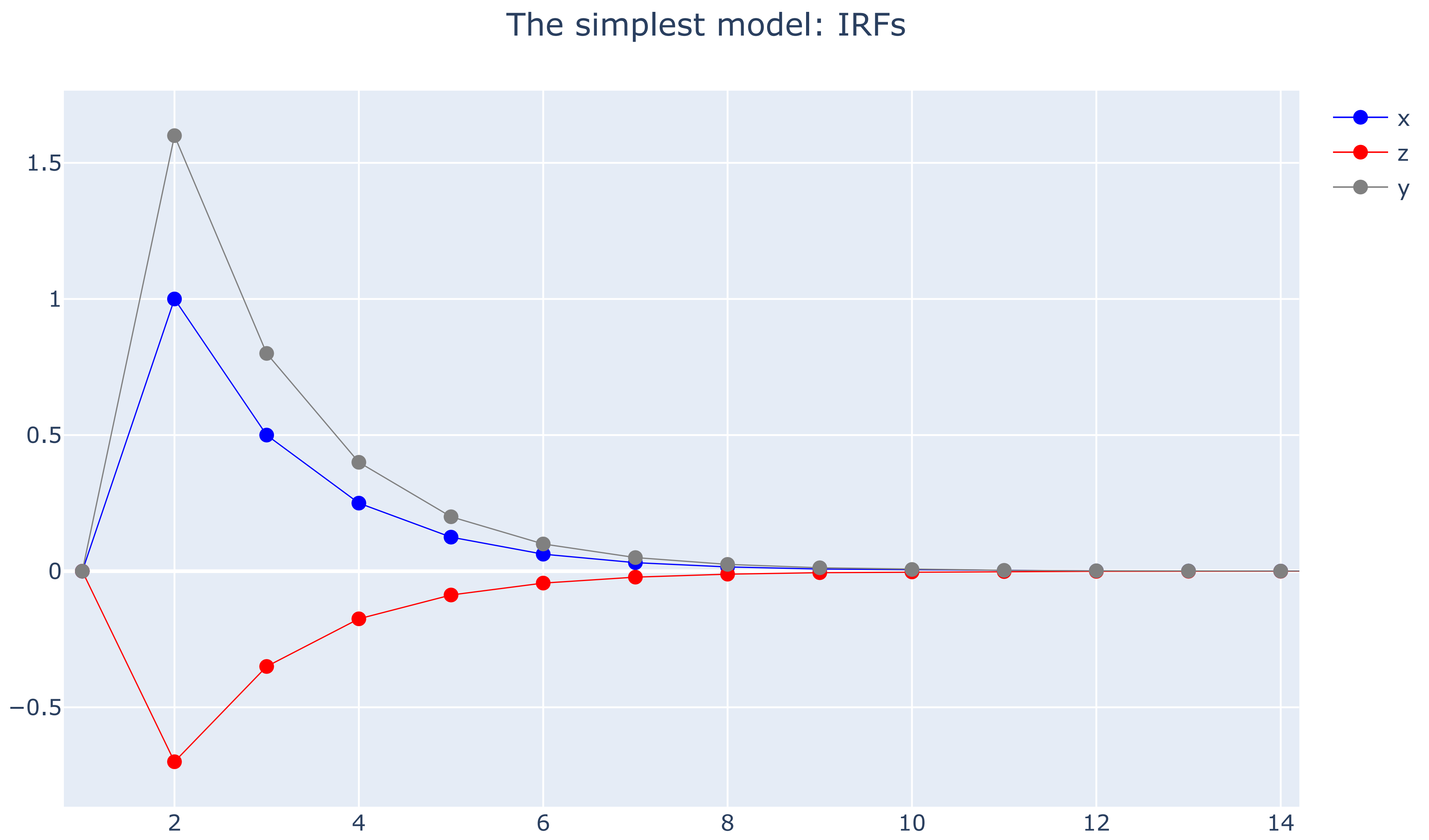
- From eq. (3), we know that: $y_t = 0 + 1.6x_t$. So

$$y_{t-3} = 1.6x_{t-3} = 1.6\,, y_{t-2} = 1.6x_{t-2}\,, \dots$$

- Once we know the values of x and y, it is immediate to calculate z.
- Assuming that $\varphi = 2.5$ and $\mu = -2$, we get:

$$z_{t-3} = 2.5x_{t-3} - 2y_{t-3} = -0.7\,, z_{t-2} = \dots$$

An image of our simple model



3. The Blanchard-Kahn conditions

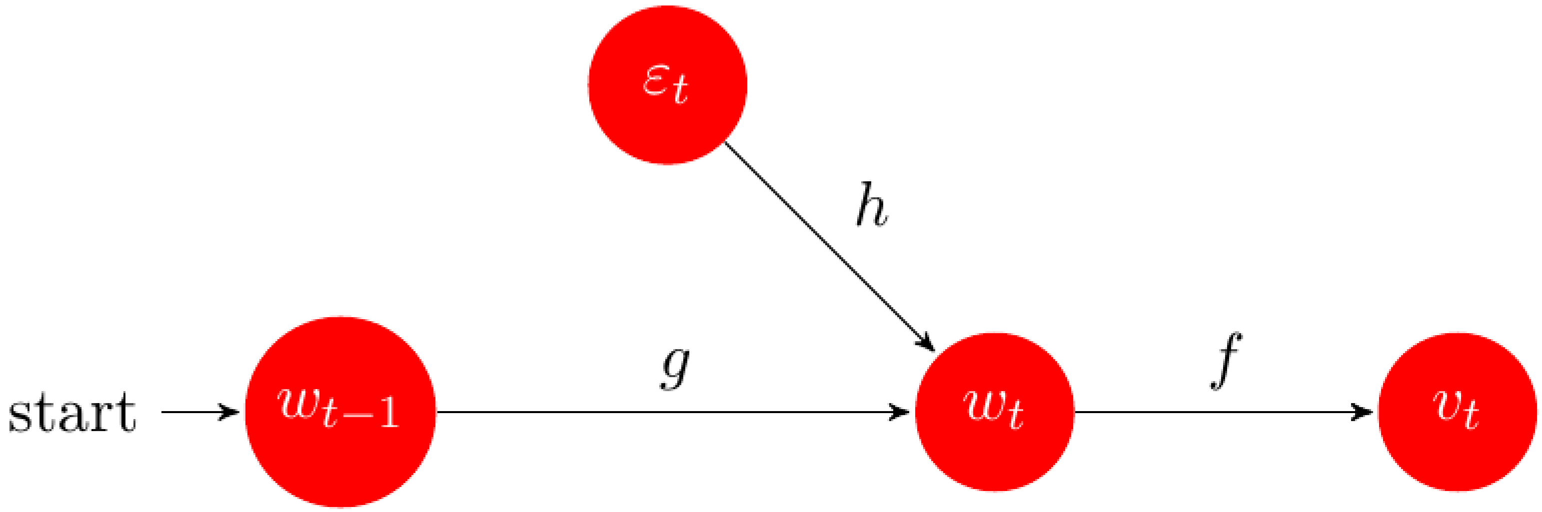
Blanchard, O. and Kahn, C. M. (1980). The solution of linear difference models under rational expectations. *Econometrica*, 48(5), 1305-1311.

More complicated models

- All models in modern macroeconomics ***can not*** be solved by pencil and paper.
- They may be non-linear.
- Their blocks may not be uncoupled, as was the case above.
- Blanchard-Kahn (1980) developed a technique that allows us to solve any **linear model**, no matter how intricate its blocks might be.
- This technique is based on the **Jordan decomposition** of square matrices.
- In this class, we do not expect students to replicate the proof; but students should understand its logic.
- It is crucial to understand what the **Blanchard-Kahn stability conditions** mean.

The strategy of solving a DSGE model

- w_t is a predetermined variable (or set of variables)
- ε_t is a random shock (or a sequence of random shocks)
- v_t is a forward-looking variable (or a set of variables)
- g and h solve the predetermined variable (or block)
- f solves the forward-looking variable (or block)



The model in state-space representation

- Write the model in state space form

$$\mathcal{A}\begin{bmatrix} w_{t+1} \\ \mathbb{E}_t v_{t+1} \end{bmatrix} = \mathcal{B}\begin{bmatrix} w_t \\ v_t \end{bmatrix} + \mathcal{C}\begin{bmatrix} \varepsilon^w_{t+1} \\ \varepsilon^v_{t+1} \end{bmatrix} + \mathcal{D} \tag{9}$$

- $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are square matrices representing the parametric structure of the model
- w_t, v_t are vectors with the endogenous variables, and ε_t is a vector of exogenous random shocks. $\mathbb{E}_t$ is the usual conditional expectations operator. $\mathcal{D}$ is a vector with constants, and for simplicity, we drop it from the model.
- Multiplying both sides of (9) by $\mathcal{A}^{-1}$, leads to:

$$\begin{bmatrix} w_{t+1} \\ \mathbb{E}_t v_{t+1} \end{bmatrix} = \underbrace{\mathcal{A}^{-1}\mathcal{B}}_{\mathcal{R}}\begin{bmatrix} w_t \\ v_t \end{bmatrix} + \underbrace{\mathcal{A}^{-1}\mathcal{C}}_{\mathcal{U}}\begin{bmatrix} \varepsilon^w_{t+1} \\ \varepsilon^v_{t+1} \end{bmatrix} \tag{10}$$

The Jordan Decomposition

- Suppose we have a square matrix $\mathcal{R}$
- The Jordan decomposition of $\mathcal{R}$ is given by:

$$\mathcal{R} = P\Lambda P^{-1}$$

- P contains as columns the eigenvectors of $\mathcal{R}$
- Λ is a diagonal matrix containing the eigenvalues of $\mathcal{R}$ in the main diagonal.
- P^{-1} is the inverse of P

Apply the Jordan Decomposition

- Our system was given by (10):

$$\begin{bmatrix} w_{t+1} \\ \mathbb{E}_t v_{t+1} \end{bmatrix} = \mathcal{R} \begin{bmatrix} w_t \\ v_t \end{bmatrix} + \mathcal{U} \begin{bmatrix} \varepsilon^w_{t+1} \\ \varepsilon^v_{t+1} \end{bmatrix} \tag{10}$$

- Apply the decomposition $\mathcal{R} = P\Lambda P^{-1}$ to (10):

$$\begin{bmatrix} w_{t+1} \\ \mathbb{E}_t v_{t+1} \end{bmatrix} = P\Lambda P^{-1} \begin{bmatrix} w_t \\ v_t \end{bmatrix} + \mathcal{U} \begin{bmatrix} \varepsilon^w_{t+1} \\ \varepsilon^v_{t+1} \end{bmatrix} \tag{11}$$

- Multiply both sides by P^{-1}

$$P^{-1} \begin{bmatrix} w_{t+1} \\ \mathbb{E}_t v_{t+1} \end{bmatrix} = \Lambda P^{-1} \begin{bmatrix} w_t \\ v_t \end{bmatrix} + \underbrace{P^{-1}\mathcal{U}}_{\mathcal{M}} \cdot \begin{bmatrix} \varepsilon^w_{t+1} \\ \varepsilon^v_{t+1} \end{bmatrix} \tag{12}$$

Matrices partition

- Let us assume that there are no shocks affecting the forward-looking block:

$$\varepsilon_t^v = 0\ ,\ \forall t$$

- Next, we apply a partition to the matrices: P^{-1}, Λ, $\mathcal{M}$.

$$\underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w_{t+1} \\ \mathbb{E}_t v_{t+1} \end{bmatrix}}_{\mathbb{E}_t \begin{bmatrix} \widetilde{w}_{t+1} \\ \tilde{v}_{t+1} \end{bmatrix}} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w_t \\ v_t \end{bmatrix}}_{\begin{bmatrix} \widetilde{w}_t \\ \tilde{v}_t \end{bmatrix}} + \underbrace{\begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{bmatrix}}_{M} \begin{bmatrix} \varepsilon_{t+1}^w \\ 0 \end{bmatrix}$$

Our transformed model looks much easier now:

$$\begin{bmatrix} \widetilde{w}_{t+1} \\ \mathbb{E}_t \tilde{v}_{t+1} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} \widetilde{w}_t \\ \tilde{v}_t \end{bmatrix} + M_{11} \cdot \varepsilon_{t+1}^w$$

The solution to the model

- Using these partitions, the solution will be given by (see detailed demonstration in **Appendix A**):

$$v_t^* = \underbrace{\left[-P_{22}^{-1} P_{21}\right]}_{f} \cdot w_t^*$$

$$w_{t+1}^* = \underbrace{\left[G^{-1} \Lambda_1 G\right]}_{g} \cdot w_t^* + \underbrace{\left[G^{-1} M_{11}\right]}_{h} \cdot \varepsilon_{t+1}$$

with $\qquad G \equiv P_{11} - P_{12}(P_{22})^{-1} P_{21}$

Partition of matrices $P^{-1}, \Lambda, \mathcal{M}$

- When solving these models, the most demanding task is to apply the correct partition to these three matrices.
- Suppose a model with 1 backward-looking variable, one static, and the third is a forward-looking variable (as the simple model above).
- The partitions should be as follows:

Partition of matrices P^{-1}, Λ, $\mathcal{M}$

2 forward-looking, 2 non-forward-looking variables

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

P11, P12, P21, P22:

$$\begin{bmatrix} P^{-1}_{1,1} & P^{-1}_{1,2} & P^{-1}_{1,3} & P^{-1}_{1,4} \\ P^{-1}_{2,1} & P^{-1}_{2,2} & P^{-1}_{2,3} & P^{-1}_{2,4} \\ P^{-1}_{3,1} & P^{-1}_{3,2} & P^{-1}_{3,3} & P^{-1}_{3,4} \\ P^{-1}_{4,1} & P^{-1}_{4,2} & P^{-1}_{4,3} & P^{-1}_{4,4} \end{bmatrix}$$

M11, M12, M21, M22:

$$\begin{bmatrix} \mathcal{M}_{1,1} & \mathcal{M}_{1,2} & \mathcal{M}_{1,3} & \mathcal{M}_{1,4} \\ \mathcal{M}_{2,1} & \mathcal{M}_{2,2} & \mathcal{M}_{2,3} & \mathcal{M}_{2,4} \\ \mathcal{M}_{3,1} & \mathcal{M}_{3,2} & \mathcal{M}_{3,3} & \mathcal{M}_{3,4} \\ \mathcal{M}_{4,1} & \mathcal{M}_{4,2} & \mathcal{M}_{4,3} & \mathcal{M}_{4,4} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon^w_1 \\ \varepsilon^w_2 \\ \varepsilon^v_1 \\ \varepsilon^v_2 \end{bmatrix}$$

The Blanchard-Kahn stability conditions

- The BK conditions are straightforward to understand
- Suppose we have a model with 5 variables:
  - 2 forward-looking
  - 2 backward-looking (or predetermined)
  - 1 contemporaneous (static)
- To secure a unique and stable solution, the matrix $\mathcal{R}$ should provide:
  - 2 eigenvalues greater than $|1|$ (forward-looking block is stable)
  - 2 eigenvalues less than $|1|$ (backward-looking block is stable)
  - 1 eigenvalue is 0 (the static variable has no dynamics of its own)
- If these conditions are violated, one of the blocks shows explosive behavior, which violates what we observe in reality.

4. Back to the ”simplest model”

Solving it with the Blanchard-Kahn method … and a computer

Prepare the model for matrix form

- The original model

$$
\begin{aligned}
x_t &= \phi + \rho x_{t-1} + \varepsilon_t^x \\
z_t &= \varphi x_t + \mu y_t \\
y_t &= \alpha + \beta \mathbb{E} y_{t+1} + \theta x_t
\end{aligned}
$$

- To write the model in matrix form, put all variables expressed at $t+1$ on the system's left side, those at t on the right side, and constants at the end.

- So, the model can be written as:

$$
\begin{aligned}
x_{t+1} &= \rho x_t + \varepsilon_{t+1}^x + \phi \\
z_{t+1} - \varphi x_{t+1} - \mu y_{t+1} &= 0 \\
\beta \mathbb{E} y_{t+1} &= -\theta x_t + y_t - \alpha
\end{aligned}
$$

The model in matrix form

$$\begin{aligned} x_{t+1} &= \rho x_t + \varepsilon^x_{t+1} + \phi \\ z_{t+1} - \varphi x_{t+1} - \mu y_{t+1} &= 0 \\ \beta \mathbb{E} y_{t+1} &= -\theta x_t + y_t - \alpha \end{aligned}$$

Detailed specification of the model

$$\begin{aligned} 1x_{t+1} + 0z_{t+1} + 0\mathbb{E}_t y_{t+1} &= \rho x_t + 0z_t + 0y_t + 1\varepsilon^x_{t+1} + 0\varepsilon^z_{t+1} + 0\varepsilon^y_{t+1} + \phi \\ -\varphi x_{t+1} + 1z_{t+1} - \mu \mathbb{E}_t y_{t+1} &= 0x_t + 0z_t + 0y_t + 0\varepsilon^x_{t+1} + 0\varepsilon^z_{t+1} + 0\varepsilon^y_{t+1} + 0 \\ 0x_{t+1} + 0z_{t+1} + \beta \mathbb{E}_t y_{t+1} &= -\theta x_t + 0z_t + 1y_t + 0\varepsilon^x_{t+1} + 0\varepsilon^z_{t+1} + 0\varepsilon^y_{t+1} - \alpha \end{aligned}$$

The model in state space representation

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\varphi & 1 & -\mu \\ 0 & 0 & \beta \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} x_{t+1} \\ z_{t+1} \\ \mathbb{E}_t y_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \rho & 0 & 0 \\ 0 & 0 & 0 \\ -\theta & 0 & 1 \end{bmatrix}}_{\mathcal{B}} \begin{bmatrix} x_t \\ z_t \\ y_t \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathcal{C}} \begin{bmatrix} \varepsilon^x_{t+1} \\ \varepsilon^z_{t+1} \\ \varepsilon^y_{t+1} \end{bmatrix} + \underbrace{\begin{bmatrix} \phi \\ 0 \\ -\alpha \end{bmatrix}}_{\mathcal{D}}$$

The state space representation passed into Julia

```
A = zeros(3,3)
B = zeros(3,3)
C = zeros(3,3)

A[1,1] = 1.0
A[2,1] =  -φ
A[2,2] = 1.0
A[2,3] =  -μ
A[3,3] =   β

B[1,1] =   ρ
B[3,1] =  -θ
B[3,3] = 1.0

C[1,1] = 1.0

D =[ϕ ; 0.0 ; -α]
```

Using the notebook "Simple_Model.jl"

- This nootebook follows step-by-step the BK approach:
  - Write the model in state-space form.
  - Check the BK stability conditions
  - Perform the matrices' partitions
  - Simulate the model's response to an isolated shock upon x_t with a magnitude of $+1$.
- And we also implement:
  - A simulation of the model's response to systematic white-noise shocks on x_t.
  - A computation of the: (i) autocorrelation function for each variable in this model, (ii) cross-correlation function, (iii) standard deviation.

Appendix A

Proof of the The Blanchard-Kahn method (not required)

The model in state-space representation

- Write the model in state space form

$$\mathcal{A}\begin{bmatrix} w_{t+1} \\ \mathbb{E}_t v_{t+1} \end{bmatrix} = \mathcal{B}\begin{bmatrix} w_t \\ v_t \end{bmatrix} + \mathcal{C}\begin{bmatrix} \varepsilon^w_{t+1} \\ \varepsilon^v_{t+1} \end{bmatrix} + \mathcal{D} \tag{A1}$$

- $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are square matrices representing the parametric structure of the model
- w_t, v_t are vectors with the endogenous variables, and ε_t is a vector of exogenous random shocks. $\mathbb{E}_t$ is the usual conditional expectations operator. $\mathcal{D}$ is a vector with constants, and for simplicity, we drop it from the model.
- Multiplying both sides of (4) by $\mathcal{A}^{-1}$, leads to:

$$\begin{bmatrix} w_{t+1} \\ \mathbb{E}_t v_{t+1} \end{bmatrix} = \underbrace{\mathcal{A}^{-1}\mathcal{B}}_{\mathcal{R}}\begin{bmatrix} w_t \\ v_t \end{bmatrix} + \underbrace{\mathcal{A}^{-1}\mathcal{C}}_{\mathcal{U}}\begin{bmatrix} \varepsilon^w_{t+1} \\ \varepsilon^v_{t+1} \end{bmatrix} \tag{A2}$$

Apply the Jordan Decomposition

- The Jordan decomposition is given by:

$$\mathcal{R} = P \Lambda P^{-1}$$

P contains as columns the eigenvectors of $\mathcal{R}$; Λ is a diagonal matrix containing the eigenvalues of $\mathcal{R}$ in the main diagonal.

- Apply the decomposition to (A2):

$$\begin{bmatrix} w_{t+1} \\ \mathbb{E}_t v_{t+1} \end{bmatrix} = P \Lambda P^{-1} \begin{bmatrix} w_t \\ v_t \end{bmatrix} + \mathcal{U} \cdot \begin{bmatrix} \varepsilon^w_{t+1} \\ \varepsilon^v_{t+1} \end{bmatrix}$$

- Multiply both sides by P^{-1}

$$P^{-1} \begin{bmatrix} w_{t+1} \\ \mathbb{E}_t v_{t+1} \end{bmatrix} = \Lambda P^{-1} \begin{bmatrix} w_t \\ v_t \end{bmatrix} + \underbrace{P^{-1} \mathcal{U}}_{\mathcal{M}} \cdot \begin{bmatrix} \varepsilon^w_{t+1} \\ \varepsilon^v_{t+1} \end{bmatrix}$$

Matrices partition

- Let us assume that there are no shocks affecting the forward-looking block:

$$\varepsilon_t = 0 \ , \ \forall t$$

- Next, we apply a partition to the matrices: P^{-1}, Λ, $\mathcal{M}$.

$$\underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w_{t+1} \\ \mathbb{E}_t v_{t+1} \end{bmatrix}}_{\mathbb{E}_t \begin{bmatrix} \widetilde{w}_{t+1} \\ \tilde{v}_{t+1} \end{bmatrix}} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w_t \\ v_t \end{bmatrix}}_{\begin{bmatrix} \widetilde{w}_t \\ \tilde{v}_t \end{bmatrix}} + \underbrace{\begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{bmatrix}}_{M} \begin{bmatrix} \varepsilon^w_{t+1} \\ 0 \end{bmatrix}$$

Our transformed model looks much easier now:

$$\begin{bmatrix} \widetilde{w}_{t+1} \\ \mathbb{E}_t \tilde{v}_{t+1} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} \widetilde{w}_t \\ \tilde{v}_t \end{bmatrix} + M_{11} \cdot \varepsilon^w_{t+1}$$

System Written as two Decoupled Blocks

Transformed model written as a set of decoupled equations:

$$\widetilde{w}_{t+1} = \Lambda_1 \cdot \widetilde{w}_t + M_{11} \cdot \varepsilon^w_{t+1} \qquad \text{(Predetermined block)}$$

$$\mathbb{E}_t \tilde{v}_{t+1} = \Lambda_2 \cdot \tilde{v}_t \qquad \text{(Forward-looking block)}$$

We can now apply our well known strategy. Iterate to:

- Solve the predetermined transformed block and get the equilibrium levels of the predetermined (backward-looking) variables:

$$\widetilde{w}^*_t$$

- Solve the forward-looking transformed block and get the equilibrium values of the forward-looking variables:

$$\tilde{v}^*_t$$

Solving the forward-looking block

- Iterating forward this block, and as the shocks to this block are 0, we get:

$$\mathbb{E}_t \tilde{v}_{t+n} = (\Lambda_2)^n \tilde{v}_t$$

- If we assume $|\Lambda_2| > 1$
- Then, the only stable solution will be

$$\tilde{v}_t^* = 0 \ , \ \forall t \tag{A3}$$

- Now, from the partition of P^{-1} and Λ, we know that

$$\tilde{v}_t^* = P_{21} \cdot w_t^* + P_{22} \cdot v_t^* \tag{A4}$$

- By equalizing (A3) and (A4), the forward-looking variables will only depend on predetermined ones.

$$v_t^* = \underbrace{\left[-P_{22}^{-1} P_{21}\right]}_{f} \cdot w_t^* \tag{A5}$$

Solving the predetermined block

- Iterating forward this block, we get

$$\widetilde{w}_{t+n} = (\Lambda_1)^n \widetilde{w}_t$$

- If we assume that

$$|\Lambda_1| < 1$$

- The process is stable, and from the partition of P^{-1} we know that:

$$\widetilde{w}_t^* = P_{11} \cdot w_t^* + P_{12} \cdot v_t^* \tag{A6}$$

- Now, inserting eq. (A5) into (A6), we can obtain:

$$\widetilde{w}_t^* = \underbrace{\left[P_{11} - P_{12} P_{22}^{-1} P_{21}\right]}_{G} \cdot w_t^* \tag{A7}$$

Solving the the predetermined block (cont.)

- As from eq. (A7) we have

$$\widetilde{w}_t^* = G \cdot w_t^* \tag{A8}$$

- Then, for $t+1$ we get:

$$\widetilde{w}_{t+1}^* = G \cdot w_{t+1}^* \tag{A9}$$

- But, as from eq. (Predetermined block) we have:

$$\widetilde{w}_{t+1} = \Lambda_1 \widetilde{w}_t + M_{11} \varepsilon_{t+1}^w \tag{A10}$$

- By mere substitution of (A8) and (A9) into (A10), we derive our final result:

$$w_{t+1}^* = \underbrace{\left[G^{-1} \Lambda_1 G\right]}_{g} w_t^* + \underbrace{\left[G^{-1} M_{11}\right]}_{h} \varepsilon_{t+1}^w \tag{A11}$$

Summarizing

1. Write down your model in state space form
2. Apply the Jordan decomposition
3. Decouple the system into two blocks
4. Make sure the eigenvalues satisfy the Blanchard-Kahn conditions
5. End up with the two fundamental results:

$$v_t^* = \underbrace{\left[-P_{22}^{-1} P_{21}\right]}_{f} \cdot w_t^*$$

$$w_{t+1}^* = \underbrace{\left[G^{-1} \Lambda_1 G\right]}_{g} \cdot w_t^* + \underbrace{\left[G^{-1} M_{11}\right]}_{h} \cdot \varepsilon_{t+1}$$

with $$G \equiv P_{11} - P_{12} (P_{22})^{-1} P_{21}$$

Readings

- This material is the application of the Blanchard-Kahn method to solve a DSGE model.
- Students **are not required to replicate** the demonstration of this method; however, they are expected to understand the logic behind this method and be able to simulate a model by using a computer and this method.
- So no required reading is really necessary. However, if one wants to have a go and see the first paper that explicitly shows how a DSGE model, without a closed form solution, can be solved and simulated, go **here** for:

> Blanchard, O. and Kahn, C. M. (1980). The solution of linear difference models under rational expectations. *Econometrica*, 48(5), 1305-1311.